Weak Wage Recovery and Precautionary Motives after a Credit Crunch

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Abstract

During the economic recovery following the financial crisis many advanced economies saw subdued wage dynamics, in spite of falling unemployment and an increasingly tight labour market. We propose a mechanism which can account for this puzzle and work against usual aggregate demand channels. In a heterogeneous agent model with incomplete markets we endogenize uninsurable idiosyncratic risk through search-and-matching (SAM) frictions in the labour market. In this setting, apart from the usual precautionary saving behaviour, households can self-insure also by settling for lower wages in order to secure a job and thereby avoid becoming borrowing constrained. This channel is especially pronounced for asset-poor agents, already close to the constraint. We introduce a credit crunch into this framework modelled as a gradual tightening of the borrowing constraint (and utilizing a continuous time approach, known as HACT). The perfect foresight transition dynamics feature falling wages despite a tightening labour market and expanding employment. As households suddenly find themselves closer to the borrowing constraint, the increased precautionary motive drives them to accept lower wages in the bargaining process, while firms respond to this by posting more vacancies, leading to a tighter labour market and falling unemployment. If the household deleveraging pressure is persistent enough after the credit crunch, it can explain the weak wage recovery in spite of already stronger aggregate demand.

keywords: precautionary savings, search and matching, credit crunch, incomplete markets, heterogeneous agents, continuous time

JEL: D52, E24, E25, G01, J31, J64

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1 Introduction

In several advanced economies the recovery from the Great Recession has been characterized by unusually weak wage growth. Even as the labor market tightened markedly and unemployment has fallen to pre-crisis lows, wage growth has failed to pick up, especially relative to previous recoveries. At the same time labor force participation rates have also fallen, productivity growth has been disappointing, while equity prices have rebounded very fast.

According to some explanations, like hysteresis or secular stagnation, the prolonged nature of the crisis or other secular trends might have changed the historic relationships between variables such as labor market slack and wage growth (somewhat akin to a flatter Phillips curve). We propose an alternative argument by combining labor market and financial frictions in an attempt to explain the above features of the recovery. While labor market developments seem especially central in driving the post-crisis recovery, the Great Recession is usually viewed as triggered by a financial shock. Therefore, the role of financial markets is potentially very important in our story.

We focus on how households' financial wealth might affect their labor market outcomes. Our hypothesis is that following a financial recession, when workers' balance sheets are still under deleveraging pressure, the utility loss associated with unemployment is larger than normally since options for consumption smoothing are more limited. Consequently, especially if they are already asset-poor, households are more desperate to keep existing or find new jobs which is why they are more likely to settle for lower wages in order to secure employment. On the other hand, this mechanism boosts the profitability of firms as their wage bill is falling which can account for rising equity prices and expanding employment.[1] Equity valuations might also be boosted by lower discount rates, as the extra saving desire pushes down real interest rates. Hence, the deleveraging pressure could also account for some of the fall (or the slow recovery) in estimated equilibrium interest rates throughout advanced economies.

To capture the above mechanism we propose a continuous time heterogeneous agent model (HACT) with incomplete financial markets and search-and-matching (SAM) frictions in the labor market, closely related to Krusell et al. (2010). Households face uninsurable idiosyncratic income risk, endogenized through SAM frictions which drive flows in and out of employment. Financial frictions are captured by the combination of a borrowing constraint and the lack of complete financial markets (i.e. no full insurance against idiosyncratic uncertainty), while labor market frictions are captured through the SAM process, whereby unemployed agents searching for a job and unfilled vacancies posted by firms cannot find each other in a seamless manner, and some jobs may be terminated for exogenous reasons. Into this framework we introduce a credit crunch, modelled as a gradual tightening of the borrowing constraint, similarly to Guerrieri and Lorenzoni (2017), and look at the perfect foresight transition dynamics in the absence of

[1] The effect on labor force participation is likely to be positive though as the deleveraging pressure shifts the labor supply curve out, even if falling wages mitigate the rise in participation.

aggregate uncertainty.

Our baseline results indicate that following a credit crunch wages fall despite a tightening labor market and expanding employment, while firm equity becomes more valuable, in line with the observed characteristics of the post-crisis recovery. In our model, apart from the usual precautionary saving behaviour, households can self-insure against idiosyncratic income risk also by settling for lower wages in order to secure a job and thereby avoid becoming borrowing constrained. This is the main transmission channel captured by the model: as households suddenly find themselves closer to the borrowing constraint, the increased precautionary motive drives them to accept lower wages in the bargaining process, while firms respond to this by posting more vacancies, leading to a tighter labour market and falling unemployment. Lower wages also mean that more of the surplus from the job stays with the firm which can boost profitability and explain higher equity prices. However, lower discount rates play a larger part in this than higher profitability: the increased saving desire boosts the asset demand of households which depresses the real interest rate. If the deleveraging pressure is persistent enough, the above responses are more prolonged.

As for the other two features of the recovery, nameley weak productivity growth and lower participation rates, our model is more silent. In its baseline version we have fully inelastic labor supply and a linear production function, under which both participation and productivity stay constant. With decreasing returns to scale, however, we are able to generate falling productivity. This can be interpreted as a shortcut to modelling damage to the supply side of the economy following a prolonged recession.

Our paper combines two main strands of the literature. On the one hand, ours is a heterogeneous agent incomplete market model with idiosyncratic income risk in the Aiyagari-Bewley-Hugett tradition. On the other hand, it also features Diamond-Mortensen-Pissarides style search-and-matching frictions in the labor market. This combination leads to endogenous idiosyncratic risk. This paper is in no respect the first to make this combination: the main point of reference is Krusell et al. (2010) to which our model is closest.[2] However, unlike Krusell et al. (2010), we spell out our model in continuous time following the work of Bardóczy (2017), which allows us to exploit analytical and numerical advantages provided by the HACT (heterogeneous agent continuous time) methodology described in Achdou et al. (2017).

Another point of departure is that our model does not have productive capital. The only savings vehicle is a fixed supply of financial assets, so the economy as a whole cannot save more. We opt for this setup in order to avoid the expansionary effects of rising investment which would be the result of increased saving desire, and which we do not find compatible with the financial crisis. In this sense, the omission of capital is a shortcut to modelling other financial frictions and nominal rigidities, under which falling aggregate demand and more binding collateral constraints could

[2]Other well-known examples include Gornemann et al. (2012), and Ravn and Sterk (2016), although these models also feature nominal rigidities.

impair investment as well.

Our focus is also different since we analyse the effects of a credit crunch in the HACT+SAM framework which, to our knowledge, has not been done yet. The modelled tightening of the borrowing constraint follows Guerrieri and Lorenzoni (2017) who work with an incomplete markets heterogeneous agent model. However, they do not endogenize idiosyncratic risk and do not model labor market frictions, which is key to our analysis. In addition, they use discrete time, while we cast the model in continuous time, incoroprating the tightening borrowing constraint according to Mellior (2016).[3] On the other hand, we do not have endogenous labor supply choice which prevents us from modelling labor force participation. Guerrieri and Lorenzoni (2017) capture self-insurance through employment via an increased participation of low productivity workers (driven by stronger precautionary motives). The corresponding composition effect is the driving force behind the drop in average productivity, average labor income and a supply-side induced recession following the credit crunch. In contrast, our model with endogenous risk and an explicit modelling of wage bargaining generates economic expansion and tightening labor markets which we view as more in line with the later stages of the recovery. Nonetheless, introducing participation choice is high on our research agenda.

At this point we would like to emphasize that ours (like the others discussed above) is a real model with fully flexible prices. By omitting nominal rigidities, we ignore aggregate demand effects and focus on the supply side of the economy. One the one hand, this allows us to identify the pure downward contribution of the precautionary channel on wage dynamics during a deleveraging process, and tells us how results are likely to change *relative* to standard New Keynesian models of aggregate demand which imply a stronger positive comovement between wages and labor market tightness. On the other hand, we necessarily fail to capture the sharp drop in economic activity in the immediate aftermath of the credit crunch which we think of as an aggregate demand driven recession (unlike in Guerrieri and Lorenzoni (2017)), but this is not the focus of our paper. To the extent that aggregate demand can start to recover even while the deleveraging is still under way, our results can be relevant.

Nevertheless, introducing nominal rigidities in a HANK+SAM fashion is high on our research agenda, as aggregate demand is a crucial part of business cycle fluctuations in general, and financial crises in particular. Examples of these models include Gornemann et al. (2012) or Ravn and Sterk (2016).[4] As the latter point out, in a HANK+SAM model countercyclical income risk may arise which can *amplify* the effects of aggregate demand shocks like a deleveraging shock. In this case precautionary channels can also reinforce aggregate demand channels instead of

[3]In short, our model can be viewed as a combination of Krusell et al. (2010) and Guerrieri and Lorenzoni (2017), but cast in continuous time based on Bardóczy (2017) and Mellior (2016), and omitting productive capital.

[4]This family of models introduce SAM labor market frictions into a general HANK (Heterogeneous Agent New Keynesian) framework which is described in Kaplan et al. (2017). HANK models usually involve aggregate uncertainty as well, which requires more advanced solution techniques as developed by Reiter (2009) and Ahn et al. (2017).

working against them.

The rest of the paper is organized as follows. Section 2 describes the theoretical model. Section 3 discusses the results. A final section concludes.

2 Model

Our model is a continuous time heterogeneous agent model with incomplete markets (HACT) and search-and-matching (SAM) frictions in the labor market. Households can either be employed or unemployed. Matches between unemployed job searchers and vacancies posted by firms are created according to a matching function, and existing jobs are separated at an exogenous rate. While firms post vacancies (i.e. decide on their labor demand) with a view to maximize profits, households do not make a labor force participation decision: they have fully inelastic labor supply (meaning they always work or search for a job). Therefore, it is the vacancy posting decision of firms which is the sole endogenous driver of the SAM-process.

Households make consumption-saving decisions in the face of idiosyncratic uncertainty. Idiosyncratic income risk is embodied in the labor market status of households (being either employed or unemployed). Due to incomplete markets households cannot fully insure themselves against this risk so individual histories of shocks matter, and will give rise to an endogenous wealth distribution (an infinite dimensional object). A standard feature of such models is the emergence of precautionary savings whereby households try to self-insure.

The combination of the heterogeneous agent incomplete market setup with labor market frictions results in idiosyncratic uncertainty being endogenized through the SAM process which drives flows in and out of employment.[5] In addition, due to labor market frictions, each match creates a surplus which is to be shared among workers and firms during a wage bargaining process. Heterogeneity in wealth creates heterogeneity also in the relative value of employment which results in an endogenous wage schedule increasing in wealth (another infinite dimensional object).

There is no productive capital in the economy and the only savings vehicles are financial assets, so the economy as a whole cannot save more or less: agents can trade risk-free government-issued bonds only among themselves. Due to SAM-frictions, however, firms have positive profits even under perfect competition which yields non-zero equity values: shares in firms constitute another financial asset. Since the value of equity can fluctuate, the supply of assets is not completely fixed.

The final good of the economy is produced with labor as the only input, and is exhasted by consumption and vacancy posting costs. The government finances unemployment benefits and

[5] One can think of this setup as a Bewley model with endogenous job finding rate, or equivalently, as a standard Diamond-Mortensen-Pissarides style search and matching model where workers can insure themselves against job loss through accummulating assets (precautionary savings).

interest payments by uniformly distributed lump sum taxes, which provide another source of insurance. There is no aggregate uncertainty. The economy does not have any nominal rigidities, so aggregate demand effects are absent.

Our model builds heavily on Bardóczy (2017), but leaves out capital accummulation and allows for a credit crunch.

2.1 Labor market

Households' income state is either employed or unemployed $s_t \in \{s^e, s^u\}$. The size of the population is normalized to one, so with u_t denoting the unemployment rate, aggregate employment is expressed as $1 - u_t$. Search-and-matching frictions in the labor market govern the dynamics of unemployment:

$$M(u_t, v_t) = u_t^{\eta}\, v_t^{1-\eta}$$

$$\lambda_t^f = \frac{M(u_t, v_t)}{v_t} = \theta_t^{-\eta} \tag{1}$$

$$\lambda_t^w = \frac{M(u_t, v_t)}{u_t} = \theta_t^{1-\eta} \tag{2}$$

$$\dot{u}_t = \sigma\,(1 - u_t) - \lambda_t^w\, u_t \tag{3}$$

where v_t is vacancies posted by firms, $\theta_t = v_t/u_t$ is the definition of labor market tightness, λ_t^f is the vacancy filling rate, λ_t^w is the job finding rate and σ is the exogenous job separation rate. The transition matrix for the *endogenous* idiosyncratic state is therefore:

$$\Lambda_t = \begin{bmatrix} -\sigma & \sigma \\ \lambda_t^w & -\lambda_t^w \end{bmatrix} \tag{4}$$

Notice that the endogeneity of the idiosyncratic income process is influenced solely through the vacancy posting decision (labor demand) of the firm, which affects labor market tightness and, in turn, job-finding rates. Since households do not make a labor force participation choice, they have no *direct* influence over the other determinant of labor market tightness, unemployment.[6]

2.2 Asset market

There are two types of financial assets in the economy: risk-free bonds issued by the government in a fixed supply B, and equity in the firm with a total market value of p. Households can also issue debt up to some borrowing constraint, however, it must be an asset for another household, so on the aggregate level the *net* supply of bonds is still B. Equity has positive value as labor market frictions result in positive firm profits: they share the surplus from a match with the worker, which drives a wedge between the marginal product of labor and the wage.

[6] Although they do have *indirect* influence on λ_t^w through the wage bargaining process which affects firms' labor demand.

In the absence of aggregate uncertainty both assets are considered risk-free, therefore a no-arbitrage condition equalizes their returns.

$$r_t = \frac{d_t + \dot{p}_t}{p_t} \tag{5}$$

where r_t is the real return on bonds, and d_t is dividends paid by the firm. For the same reason, the household is indifferent between the two assets which is why we do not model portfolio choice but only the total value of a household's asset holdings defined as

$$a_t^i = \vartheta_t^i (B + p_t)$$

where ϑ_t^i is the share of a particular household i from total assets.

In the presence of idiosyncratic uncertainty the above setup means that financial markets are incomplete and full insurance against all contingencies is not possible.

2.3 Households

There is a continuum of households $i \in [0, 1]$, who make a consumption-saving choice in the face of idiosyncratic income risk. Their income state is either employed or unemployed $s_t \in \{s^e, s^u\}$ which is governed by a Poisson process determined by the SAM frictions as in Λ_t. Households derive no utility from leisure, so they have no endogenous labors supply (participation) choice: if offered a job, they work.[7] If employed, workers earn a wage according to a wage schedule $\omega_t(a_t)$ which depends on their wealth. Unemployed workers get unemployment benefits h from the government. All households pay lump sum taxes T_t to the government. In making their consumption-saving decision, households are subject to an exogenous borrowing constraint, meaning that their total assets cannot go below $\underline{a}$ (or the natural borrowing limit, if it is stricter).

The sequential formulation of a household's problem is the following:

$$\begin{aligned} W(a_0, s_0) = \max_{c_t, \dot{a}_t} \quad & \mathbb{E}_0 \int_0^\infty e^{-\rho t}\, u(c_t)\, dt \\ & \dot{a}_t = y_t(a_t, s_t) + r_t\, a_t - c_t \\ & a_t \geq \max\left\{\underline{a};\ -\frac{h}{r}\right\} \\ & s_t \in \{s^e, s^u\} \sim \text{Poisson } (\Lambda_t) \\ y_t(a_t, s_t) = & \begin{cases} \omega_t(a_t) - T_t & \text{if } s_t = s^e \\ h - T_t & \text{if } s_t = s^u \end{cases} \end{aligned}$$

[7]This also means that the law of motion for unemployment (3) is not a constraint in their problem. Current employment is a state variable in SAM-models, as it is determined by matching frictions. However, its next period value could be influenced through searching more intensively and increasing participation today, subject to (3). Without participation choice households cannot influence next period's chances of getting employment by increasing their search/unemployment today.

where ρ is the personal discount rate. The change in assets $\dot{a}_t$ is interpreted as the flow savings of the household, which is pinned down buy the budget constraint.[8]

In continuous time it is convenient to write the above sequential problem recursively in the form of Hamilton-Jacobi-Bellman equations and boundary conditions. In doing so, we follow closely the HACT methodology as explained in Achdou et al. (2017), and as applied by Bardóczy (2017) to models with SAM frictions. [9]

$$\rho\, W_t(a,s^e) = \max_c \Big\{ u(c) + \partial_a W_t(a,s^e)\, \underbrace{\big[\omega_t(a) - T_t + r_t\, a - c\big]}_{\dot{a}_t} + \sigma\big[W_t(a,s^u) - W_t(a,s^e)\big] + \partial_t W_t(a,s^e) \Big\} \tag{6}$$

$$\rho\, W_t(a,s^u) = \max_c \Big\{ u(c) + \partial_a W_t(a,s^u)\, \underbrace{\big[h - T_t + r_t\, a - c\big]}_{\dot{a}_t} + \lambda_t^w\big[W_t(a,s^e) - W_t(a,s^u)\big] + \partial_t W_t(a,s^u) \Big\} \tag{7}$$

$$\begin{aligned}
&\partial_a W_t(\underline{a}, s^e) \geq u'\Big(\omega_t(\underline{a}) + r_t\, \underline{a}\Big) \\
&\partial_a W_t(\underline{a}, s^u) \geq u'\Big(h + r_t\, \underline{a}\Big)
\end{aligned}$$

where $W_t(a,s)$ is the value function of households over the state space (a,s). The relevant state variables for the household in making their consumption/saving decision are the income state (being employed or unemployed) and financial wealth: (a,s). The FOC with respect to consumption and the budget constraint give us the optimal consumption and savings policies *as a function of* the value function's derivative (which is itself a function of the state variables).

$$c_t(a,s) = u'^{-1}\Big(\partial_a W_t(a,s)\Big) \tag{8}$$

$$\dot{a}_t(a,s) = y_t(a,s) + r_t\, a - c_t(a,s) \tag{9}$$

[8] The reader can convince themselves that the return on total assets a_t is the same as on bonds due to the no-arbitrage condition with equity:

$$\begin{aligned}
a_t &\equiv b_t + p_t \\
a_t^+ &= (1 + r_t \Delta t) b_t + p_t + \Delta p + d_t \Delta t \\
a_t^+ - a_t &= r_t \Delta t\, b_t + \Delta p + d_t \Delta t \\
\frac{a_t^+ - a_t}{\Delta t} &= r_t\, b_t + \underbrace{\frac{\Delta p}{\Delta t} + d_t}_{r_t\ p_t} \;=\; r_t\ a_t
\end{aligned}$$

[9] Notice that the HJB equations are just the special versions for each income state s^e, s^u of the more general HJB formulation:

$$\rho\, W_t(a,s) = \max_c \Big\{ u(c) + \partial_a W_t(a,s)\, \underbrace{\big[y_t(a,s) + (r_t - \delta)a - c\big]}_{\dot{a}_t} + \sum_{s' \neq s} \lambda_{ss'} \big[W_t(a,s') - W_t(a,s)\big] + \partial_t W_t(a,s) \Big\}$$

$$\partial_a W_t(\underline{a}, s) \geq u'\Big(y_t(\underline{a}, s) + (r_t - \delta)\underline{a}\Big)$$

For a precise and detailed derivation of the HJB equations in continuous time problems we refer the interested reader to Achdou et al. (2017).

One of the advantages of using continuous time (apart from numerical and computational efficiencies) is that the borrowing constraint collapses into simple boundary conditions as above (because unlike in discrete time, it applies to a *state* variable rather than a control). More importantly, the first-order conditions (8) hold with equality *everywhere* in the state space (i.e. even at the borrowing constraint) – unlike in discrete time, where the Euler-equation is an inequality and will be slack whenever the borrowing constraint binds. In addition, the FOC is *static* which allows us to directly solve for the optimal consumption choice. In discrete time the FOC involves tomorrow's asset level (a choice variable) which defines optimal consumption only implicitly, requiring costly root finding methods to solve it.[10]

2.3.1 Distribution

Due to incomplete markets different histories of idiosyncratic income shocks will lead to different asset levels for individual households, giving rise to a non-degenerate wealth distribution, an inifinite dimensional object which is an important component of the economy's aggregate state. Let $g_t(a,s)$ denote the density of the joint distribution of households over the asset-income state spce. Given the optimal consumption/saving choices $\dot{a}_t(a,s)$ from (9), which govern movements along the asset dimension, and the SAM-determined Poisson process for income Λ_t from (4), the dynamics of the distribution are described by the Kolmogorov Forward Equations: [11]

$$\partial_t g_t(a,s^e) = -\partial_a\Big[\dot{a}_t(a,s^e)\; g_t(a,s^e)\Big] - \sigma\; g_t(a,s^e) + \lambda_t^w\; g_t(a,s^u) \tag{10}$$

$$\partial_t g_t(a,s^u) = -\partial_a\Big[\dot{a}_t(a,s^u)\; g_t(a,s^u)\Big] - \lambda_t^w\; g_t(a,s^u) + \sigma\; g_t(a,s^e) \tag{11}$$

The density $g_t(a,s)$ naturally integrates to one, and since the population size is also normalized to one, the mass of employed and unemployed households give use the employment and unemployment rates, respectively.

$$\sum_{s\in\{s^e,s^u\}}\int_{\underline{a}}^{\infty} g_t(a,s)\;da = \underbrace{\int_{\underline{a}}^{\infty} g_t(a,s^e)\;da}_{1-u_t} + \underbrace{\int_{\underline{a}}^{\infty} g_t(a,s^u)\;da}_{u_t} = 1$$

[10] To see these differences, it might be helpful to compare continuous time and discrete time FOCs:

$$\begin{aligned} u'(c) &= W_a(a,y_j) \\ u'(c) &\geq \beta\;\mathbb{E}_{y'}\;W_a(a',y') \\ &\geq \beta\;\sum_{k=1}^{J} Pr(y_k|y_j)\;W_a\Big(\underbrace{y_j+(1+r)a-c}_{a'(a,y_j)},\;y_k\Big) \end{aligned}$$

[11] For a detailed derivation of the KFE equation we again refer to Achdou et al. (2017). The general KFE is:

$$\partial_t g_t(a,s) = -\partial_a\Big[\dot{a}_t(a,s)\;g_t(a,s)\Big] - g_t(a,s)\sum_{s'}\lambda_{ss'} + \sum_{s'}\lambda_{s's}\;g_t(a,s')$$

2.4 Firms

Firms produce the output of the economy using labor as the only input according to the following production technology:

$$\begin{aligned} z_t\, F(N_t) &= z_t\, N_t^{1-\alpha} \\ \frac{z_t\, F(N_t)}{N_t} &= z_t\, N_t^{-\alpha} = z_t\,(1-u_t)^{-\alpha} \end{aligned}$$

where z_t denotes TFP and the last line expresses per-capita production, i.e. the output corresponding to a single job, if matched with a worker. Within a job there is no intensive margin: the worker either works full hours or nothing. $\alpha = 0$ corresponds to a linear production technology, and CRS production. With $\alpha > 0$ we have decreasing returns to scale and also diminishing marginal product of labor, which means that the marginal product of an additional job will depend on the aggregate level of employment.

Firms create jobs by posting vacancies v_t at a fixed cost of ξ. Each job commands a wage $\omega_t(a)$ which depends on the wealth of the worker. Due to SAM frictions the firm solves a dynamic problem: current employment is a state variable, so it is only future employment which can be influenced by current vacancy posting decisions, subject to the dynamics imposed by SAM frictions in (3). Therefore, the labor demand choice is implicit in the vacancy posting decision.

We can write up the firm's dynamic problem recursively, using HJB equations involving value functions for a *single* filled job $J_t(a)$ and an unfilled vacancy V_t. Profits are discounted at a rate r_t (which is the relevant alternative cost for households who own the firm). In their profit maximiziation problem firms take into account that the state can change in the next instant, i.e. the job might be separated at a rate σ and an unfilled vacancy might get filled at a rate λ_t^f as well as the employed worker's asset position might change at a rate of their savings policy $\dot{a}_t(a, s^e)$ which would alter the wage payable to them. The resulting HJB equations are:

$$\begin{aligned} r_t\, J_t(a) &= \left[z_t\,(1-u_t)^{-\alpha} - \omega_t(a)\right] + \partial_a J_t(a)\; \dot{a}_t(a, s^e) + \sigma\Big[V_t - J_t(a)\Big] + \partial_t J_t(a) \\ r_t\, V_t &= -\xi + \lambda_t^f \int_{\underline{a}}^{\infty} J_t(a)\, \frac{g_t(a, s^u)}{u_t}\, da \end{aligned}$$

Due to free entry the value of opening a new vacancy must be zero in equilibrium. Therefore the labor demand decision of the firm is embedded in the condition that

$$V_t = 0 \tag{12}$$

i.e. the firm will post vacancies until their value drops to zero, which fills in the role of a FOC with respect to v_t. Plugging this optimality condition back into the HJBs, we get:[12]

$$(\sigma + r_t)\, J_t(a) = \left[z_t\,(1-u_t)^{-\alpha} - \omega_t(a)\right] + \partial_a J_t(a)\ \dot{a}_t(a,s^e) + \partial_t J_t(a) \tag{13}$$

$$\xi = \lambda_t^f \int_{\underline{a}}^{\infty} J_t(a)\,\frac{g_t(a,s^u)}{u_t}\,da \tag{14}$$

Due to matching frictions there will be a wedge between the marginal product of labor and the real wage. Therefore, even in the presence of free entry the firms has positive profits which are paid out as dividend to the household. If $\pi_t(a)$ denotes the ex-vacancy profits of the firm, corresponding to a particular job held by a worker with asset level a and earning wage $\omega(a)$, then aggregate dividends d_t are determined as:

$$\pi_t(a) = z_t\,(1-u_t)^{-\alpha} - \omega_t(a) \tag{15}$$

$$d_t = \int_{\underline{a}}^{\infty} \pi_t(a)\ g_t(a,s^e)\ da - \xi v_t \tag{16}$$

2.5 Wage setting

The surplus from a match is shared between the worker and the firm according to some bargaining process, with β denoting the bargaining power of the worker. We explore two types of bargaining. Under Nash-bargaining the wage schedule is the solution to the problem

$$\omega_t(a) = \arg\max_{w} \left[\widetilde{W}_t(a,s^e,w) - W(a,s^u)\right]^{\beta}\left[\widetilde{J}(a,w)\right]^{1-\beta} \tag{17}$$

Alternatively, we can have egalitarian bargaining, where the surplus is shared according to

$$(1-\beta)\Big[W_t(a,s^e) - W_t(a,s^u)\Big] = \beta\, J_t(a)$$

As Bardóczy (2017) shows, in both cases continuous time allows for a closed form solution for the wage schedule.[13]

The main point here is the emergence of a wage schedule $\omega_t(a)$ which depends positively on the wealth of the worker – as opposed to a single wage being paid to every worker, which is the case in standard SAM models with complete markets. This is due to the fact that the relative value of the worker's outside option, i.e. of turning down the job offer and staying unemployed, depends on their wealth. With less assets to rely on to smooth consumption (or equivalently being closer to the borrowing constraint) becoming/staying unemployed makes a much bigger difference than with sufficient wealth. Therefore, the worker is more eager for the job and is

[12]Notice that the HJB equation for jobs (13) is not a maximization (as the only decision with respect to vacancies has already been taken): it is just an expression to derive the value coming from a filled job $J_t(a)$ which can later be used in the wage bargaining process.

[13]For the details of derivation, we refer to Bardóczy (2017). $\omega_t(a)$ is extracted from $W_t(a,s^e)$ and $J_t(a)$ when we solve the above problems. The "tilde" value functions include an arbitrary wage w, which coincide with the actual value function when using the optimal wage schedule $w = \omega(a)$. So $\widetilde{W}_t(a,s^e,\omega(a)) = W_t(a,s^e)$, while $\widetilde{J}_t(a,\omega(a)) = J_t(a)$, and $W_t(a,s^u)$ does not depend on the wage.

willing to accept lower wages in order to avoid getting closer to the borrowing constraint. In this sense, accepting lower wages in order to secure employment is a form of self-insurance against idiosyncratic uncertainty: on the one hand, it is a substitute for precautionary savings, but on the other hand it is also a means to secure higher income which in turn allows for more precautionary savings. This mechanism is key in our model, which drives wage dynamics following a tightening of the borrowing constraint.

2.6 Government

The government sustains a stable debt of B. It finances unemployment benefits and interest payments on its debt by collecting lump-sum taxes from the households.[14] The government's budget constraint is therefore:

$$r_t\, B = T_t - u_t\, h \tag{18}$$

2.7 Market clearing

Asset market clearing means that demand for financial assets from the households equals the total supply of financial assets which are a fixed supply B of government bonds and equity in the firms, valued at p_t:

$$\sum_{s\in\{s^e,s^u\}} \int_{\underline{a}}^{\infty} a\; g_t(a,s)\; da = B + p_t \tag{19}$$

Equilibrium in the asset market will be achieved by the adjustment of the real interest rate r_t which influences saving decisions and therefore the asset demand of households (reflected in the asset distribution $g_t(a,s)$).[15]

Labor market clearing is already implicit in the formulation of the model, as laid out above.

[14] We maintain the possibility of no government, in which case $u_t\, h$ is to be interpreted as home production by unemployed agents, while $r_t\, B$ is interest income earned on foreign assets (which is equivalent to the trade deficit).

[15] In fact, a changing interest rate also affects asset supply through influencing equity valuation as the present value of future dividends change.

The goods market should automatically clear by Walras' law, once the other markets clear.[16]

$$C_t + \xi v_t = \underbrace{z_t\,(1-u_t)^{-\alpha}}_{Y_t}$$

This is the resource constraint of the economy which shows that final output is exhausted by consumption and vacancy posting costs.[17]

2.8 Equilibrium

Equations (1) to (19) (together with the boundary conditions incorporating the borrowing constraint) describe the equilibrium of the model . The equilibrium consists of 19 variables which are:

- a set of quantities $u_t, v_t, T_t, \lambda_t^f, \lambda_t^w, d_t, \pi_t(a)$ – (also defining tightness $\theta_t = v_t/u_t$),
- a set of value functions $W_t(a,s^e), W_t(a,s^u), J_t(a), V$,
- a set of policy functions $c_t(a,s), \dot{a}_t(a,s)$,
- distributions over assets and employment $g_t(a,s^e), g_t(a,s^u)$
- a set of prices $r_t, p_t, \omega_t(a)$
- transition probabilities between employment states Λ_t

[16]This can be verified by aggregating the individual budget constraints of households and the government.

$$c_t(a,s) + \dot{a}_t(a,s) = y_t(a,s) + r_t\, a$$

$$C_t + [\dot{B} + \dot{p}_t] = \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da + h\;\underbrace{\int_{\underline{a}}^{\infty} g_t(a,s^u)\; da}_{u_t}\; - T_t \;+\; r_t\;(B + p_t)$$

$$C_t = \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da + \underbrace{u_t\, h\; - T_t + r_t B}_{=0 \text{ by (18)}} + \underbrace{r_t\; p_t - \dot{p}_t}_{d_t \text{ by (5)}}$$

$$C_t = \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da + \underbrace{\int_{\underline{a}}^{\infty} \pi_t(a)\; g_t(a,s^e)\; da - \xi v_t}_{d_t \text{ by (16)}} =$$

$$= \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da + \int_{\underline{a}}^{\infty} \underbrace{\Big[z_t(1-u_t)^{-\alpha} - \omega_t(a)\Big]}_{\pi_t(a) \text{ by (15)}}\; g_t(a,s^e)\; da - \xi v_t$$

$$= \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da + \Big[z_t(1-u_t)^{-\alpha}\Big] \underbrace{\int_{\underline{a}}^{\infty} g_t(a,s^e)\; da}_{1-u_t} - \int_{\underline{a}}^{\infty} \omega_t(a)\; g_t(a,s^e)\; da - \xi v_t$$

$$= z_t(1-u_t)^{1-\alpha} - \xi v_t$$

[17]In the case of no government, final output would include home production in addition to private firm production. There would also be a trade deficit term as the interest income earned on foreign assets $r_t\, B$ would allow for higher domestic absorbtion than total output.

3 Results

3.1 Solution method

We model the credit crunch as a gradual tightening in the effective borrowing constraint, which essentially follows Guerrieri and Lorenzoni (2017). A slight technical difference in our case, due to computational convenience, is that the terminal borrowing constraint immediately jumps from $\underline{a} = a_1$ to its new value $\underline{a} = a_x$, but households who find themselves in the newly inadmissible region $a_1 \leq a < a_x$ are required to save their way towards the tighter borrowing limit only gradually (for those already above a_x, the new constraint is immediately binding). This is achieved by imposing positive savings in this region of *at least* Δa, provided that the non-negativity of consumption is not violated.[18,19]

$$\dot{a}_t(a,s) \geq \begin{cases} \min\left\{\Delta a,\ y_t(a,s) + r_t\, a\right\} & \text{for } \forall\ a_1 \leq a < a_x \\ \min\left\{0,\ y_t(a,s) + r_t\, a\right\} & \text{for } a = a_x \end{cases}$$

We ignore aggregate uncertainty, and we look at the perfect foresight transition dynamics between the two stationary equilibria: the initial one featuring the original borrowing constraint a_1, while the terminal one having a tighter borrowing constraint a_x. During the transition we impose the above rule for positive savings in the inadmissible region. This is similar to an "MIT type" of unexpected shock, which can be interpreted as the announcement of the new constraints.

In solving our HACT model, we follow Achdou et al. (2017) and use their finite difference scheme of upwinding to discretize the Hamilton-Jacobi-Bellman and Kolmogorov Forward equations. This procedure is very convenient in continuous time as the discretized versions of both the HJB and KF equations will feature the same $\mathbf{A}_t$ matrix which describes transition rates within the (discretized) idiosyncratic state space. Continuous time means that this transition matrix is extremely sparse (only neighboring states can be reached within an instant of time) which allows for speedy computations. For details of this method we refer the reader to Achdou et al. (2017).

[18] Over time this condition will make sure that no households remain in $a_1 \leq a < a_x$ as they save themselves away from this region of state space. However, following Mellior (2016), we can completely guarantee this by imposing a stricter condition in the final period or terminal stationary equilibrium. This requires that in case there are still some households in the inadmissible region, they immediately jump to the new constraint by saving at least $\dot{a}_T(a,s) \geq a_x - a$ instead of Δa.

[19] Notice that these conditions translate into appending the boundary conditions accompanying the HJB equations of the household as follows:

$$\partial_a W_t(a,s) \geq \begin{cases} u'\Big(\max\left\{y_t(a,s) + r_t\, a - \Delta a,\ 0\right\}\Big) & \text{for } \forall\ a_1 \leq a < a_x \\ u'\Big(\max\left\{y_t(a_x,s) + r_t\, a_x,\ 0\right\}\Big) & \text{for } a = a_x \end{cases}$$

We introduce SAM frictions into the HACT framework based on Bardóczy (2017) who implements Krusell et al. (2010) in continuous time. We incorporate the credit crunch into the HACT setting by appending the state constraints and boundary conditions as inspired by Mellior (2016). The details of our numerical algorithm can be found in the Appendix to this paper.

3.2 Calibration

`The model is not properly calibrated yet. The values below are for illustrative purposes.`

Parameters			
γ	1.00	χ	1.10
ρ	0.05	σ	0.15
α	0.00	η	0.72
z	1.00	β	0.72
B	0.50	h	0.30
		ξ	0.199
Steady states	initial		terminal
$\mathbb{E}[\omega(a)]$	0.9576	$\mathbb{E}[\omega(a)]$	0.9576
u	0.1118	u	0.1113
θ	1.3294	θ	1.3537
p	0.1866	p	0.1874
r	0.0435	r	0.0412

Table 1: Parameters and selected steady state values

The utility function is CRRA, where $\gamma = 1$ corresponds to log utility $u(c_t) = \log c_t$.

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

Time is continuous and $t = 1$ in the model corresponds to one year. The personal discount factor of $\rho = 0.05$ corresponds to an annual 5% equilibrium real interest rate under complete markets. With incomplete markets the precautionary saving motive depresses it to 4.35%. And even further with the tightening of the borrowing constraint to 4.12%.

In the baseline parametrization we use egalitarian bargaining (Nash bargaining results in a flatter wage schedule). The Hosios condition is satisfied as the worker's bargaining power β equals the matching elasticity with respect to unemployment η. Production technology is linear and constant returns to scale ($\alpha = 0$). `The unemployment rate is not matched to data yet.`

3.3 Baseline credit crunch

In our baseline scenario we look at the effects of tightening the borrowing constraint from $a_1 = -2$ to $a_x = -1.44$ with gradual steps of $\Delta a = 0.07$ required (i.e. this is a tightening of $x = 8$ gridpoints). As Figure 1 illustrates, this causes a rightward shift in the asset demand curve $\sum_{s\in\{s^e,s^u\}} \int_{\underline{a}}^{\infty} a\ g_t(a,s;r_t)\,da$, reflecting an increased saving desire by households.[20] One the one hand, the stricter borrowing constraint directly forces asset-poor households to save more as they need to gradually deleverage and get out of the newly inadmissible asset region $a_1 \leq a < a_x$. On the other hand, all households suddenly find themselves closer to the borrowing constraint which increases precautinary saving motives throughout the wealth distribution, but especially for poorer households.

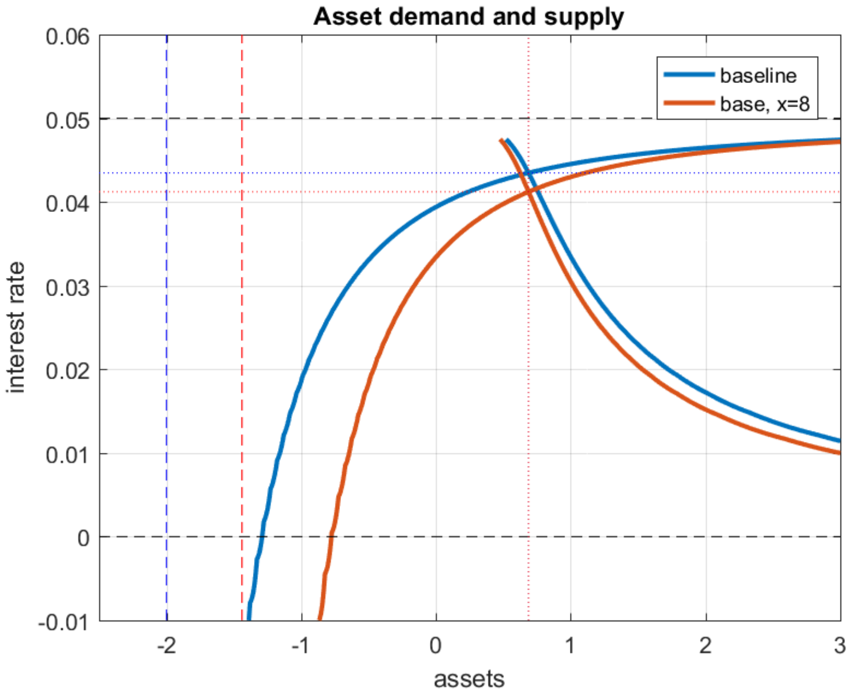


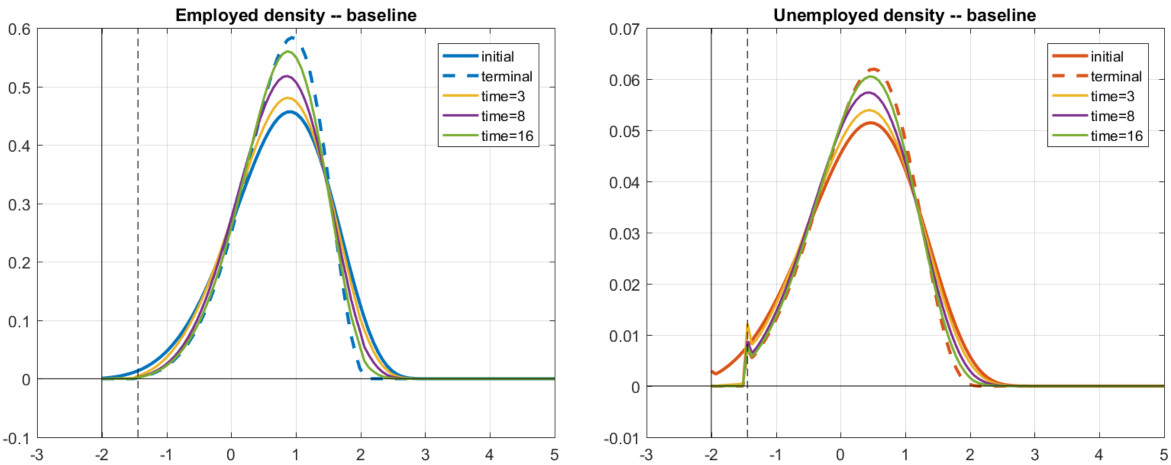
Figure 1: Asset demand $\sum_s \int_{\underline{a}}^{\infty} a\ g_t(a,s;r_t)\,da$ and asset supply $B + \frac{d_t+\dot{p}_t}{r_t}$. Blue lines correspond to the initial equilibrium, while red lines depict the terminal equilibrium after the credit crunch. Vertical dashed lines show the tightening in the borrowing constraint. The upper black dashed line is the interest rate (and asset demand) under complete markets, ρ. Dotted lines trace out equilibrium interest rates and assets.

[20] Note that household asset demand, i.e. saving desire is a positive function of the interest rate for intertemporal reasons. In the formula for aggregate asset demand the interest rate enters through its effect on the wealth distribution $g_t(a,s;r_t)$.

Precautionary saving motives are due to incomplete markets and are captured by the asset demand curve's distance from the complete market benchmark (a vertical line at the borrowing constraint and a horoznital line at the personal discount rate ρ). With sufficient amount of assets above the borrowing constraint asset demand converges to its complete markets version. The higher degree of idiosyncratic risk there is, the further away is the curve from the complete market case. The closer we are to the borrowing constraint, the more the increased precautionary saving motive depresses the interest rate below ρ.

Despite the rise in aggregate asset demand, with a fixed supply B of government bonds the economy as a whole cannot actually save more, so the increased saving desire needs to be discouraged by a lower real interest rate which is where asset market equilibrium is restored. However, asset supply is not completely fixed since the value of equity is a negative function of the discount rate $p_t = \frac{d_t + \dot{p}_t}{r_t}$, which would mitigate the asset shortage and the fall in r_t. It turns out though, that firm profitability actually worsens a little bit (discussed later), shifting asset supply to the left and counteracting most of the discount rate effect. Equity prices still rise, but not by much, which is why aggregate savings are essentially unchanged.

Looking at the distribution of assets instead of aggregate measures, in Figure 2 we see that the direct deleveraging pressure and rise in precautionary saving motives are strongest for asset-poor households who are relatively close to the borrowing constraint: the lower segments of the net wealth distribution are emptied out gradually as those households rebuild their balance sheets. Since the economy as whole cannot increase its savings this must be offset by dissaving from the part of richer households, and it is exactly what we see in the right tail of the distribution: wealthy agents are incentivized to reduce their asset holdings by a lower equilibrium interest rate. For the above reasons, instead of seeing a rightward shift in the asset distribution, it rather becomes more concentrated following the credit crunch.



(a) Wealth distribution of employed

(b) Wealth distribution of unemployed

Figure 2: Evolution of wealth distributions after a credit crunch

Under SAM frictions these asset market developments have an effect on the labor market as well. As discussed in the model description in Section 2.5, the wage schedule $\omega_t(a)$ arising from wage bargaining is a positive function of wealth. More precisely, it is a positive function of the *distance* from the borrowing constraint: it reflects the relative value of employment to the outside option (unemployment) which is much larger when there is a more limited room for consumption smoothing. As households suddenly find themselves closer to a tighter borrowing constraint, they become keener to secure a job and are willing to accept lower wages as a form of self-insurance. This channel works parrallel to and is also a substitute for precautionary savings,

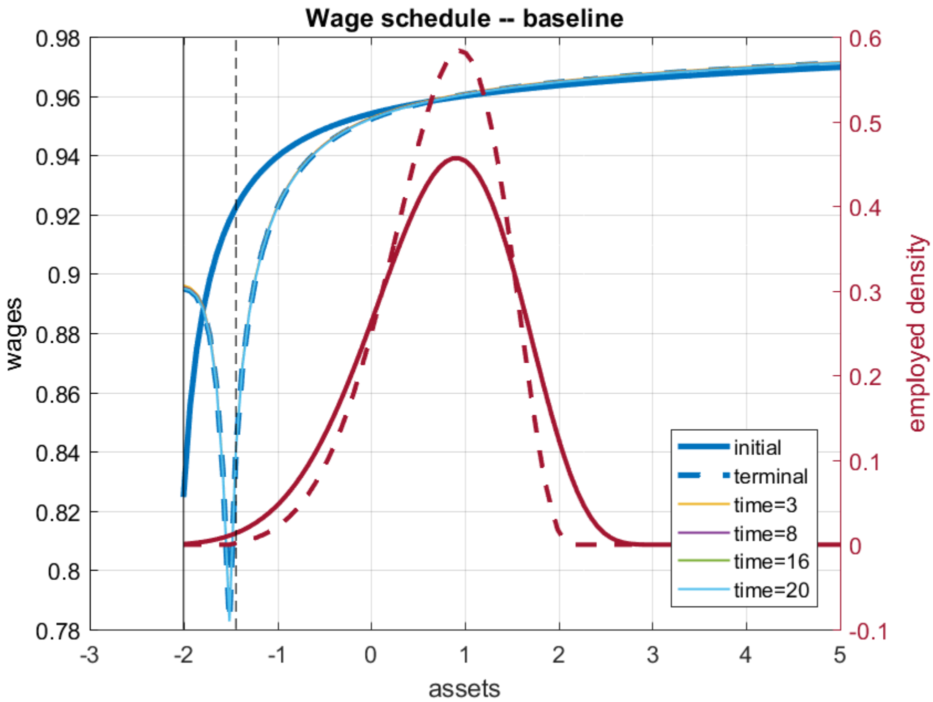


Figure 3: Evolution of the wage schedule after a credit crunch. Deep purple solid and dashed lines depict the initial and terminal density function for employed workers (rhs), respectively.

and is driven by the same factors.

In line with the above argument, and as Figure 3 demonstrates, the wage schedule shifts down for poorer workers, immediately reaching its new steady state.[21] During the transition while there are still some mass of workers in these regions, this puts downward pressure on the average wage as well, but due to the deleveraging pressure there will eventually be fewer agents here and the average wage will recover.

The latter point can be seen in the top left panel of Figure 4. The average wage falls on impact after the credit crunch, but as the deleveraging process goes on and agents rebuild their balance sheets, they also move up along the suddenly lower wage schedule, leading to a recovery in the average wage. All along the transition, however, wages stay lower than they would have in the absence of the credit crunch, and this difference is entirely due to the interaction of increased precautionary motives and labor market frictions, the main channel in our argument for explaining weak wages. By chaining this impulse response to one arising from a standard model of aggregate demand without this channel, we can demonstrate why we see both a sharper

[21] The kink in the new wage schedule at the new borrowing constraint is due to the fact that below this level of assets we require at least a constant amount of saving Δa. This causes a kink in the consumptions policy function of unemployed agents who would have had negative savings in this region. This carries over to the consumptions policy of employed agents (who are interested in their relative position to the unemployed state) which in turn affects their wage schedule. If instead the minimum deleveraging requirement would be an increasing function of the shortfall from the new tighter borrowing constraint, this kink could be smoothed, but we opted for a constant Δa for everybody to allow for a more gradual adjustment for more "underwater" households. See Appendix B

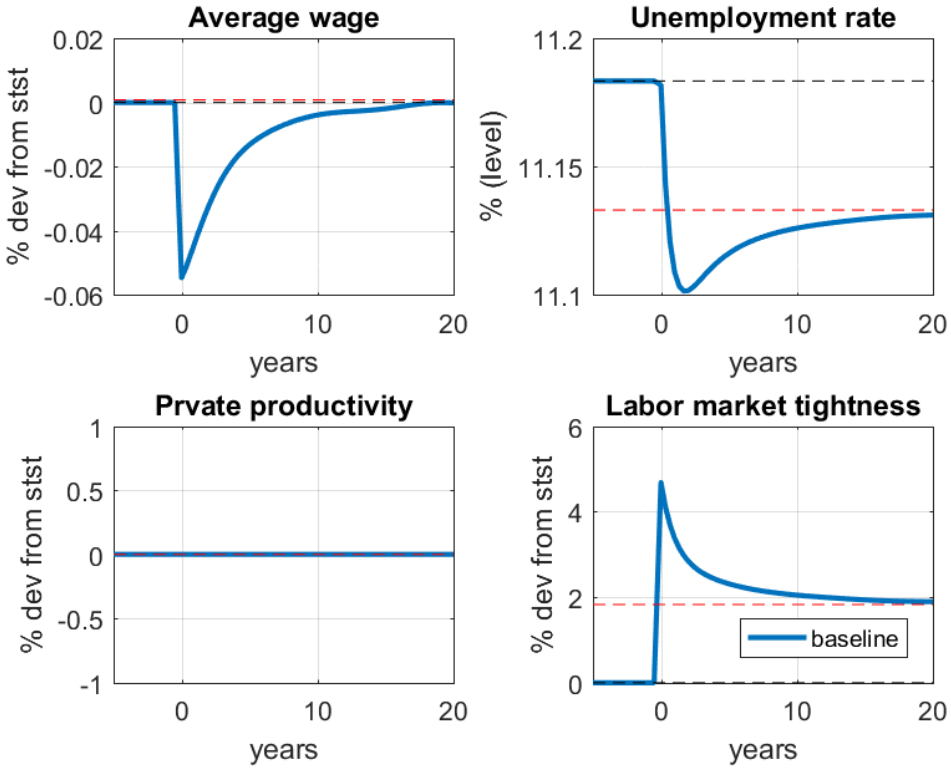


Figure 4: Impulse responses after a tightening of the borrowing constraint – labor market

fall in wages on impact as well as a weaker wage recovery.

Figure 4 also shows that falling wages occur against the backdrop of falling unemployment and a tightening labor market, in line with the puzzling characteristics of the post-crisis recovery outlined in the Introduction. Lower wages prompt firms to open more vacancies and the tightening labor market leads to more matches, higher job finding rates and therefore falling unemployment. The combination of lower wages and higher employment might point towards a positive labor supply shock, which is true in the sense that the change comes from the household's side as they are more desperate for jobs. But recall, that our model does not have endogenous labor supply choice and household decisions are only reflected through the wage bargaining process. In addition, a classic labor supply shock would not result in a tightening labor market.

Observing Figure 4 we can notice that while the wage reverts back to its original steady state, labor market tightness and employment settle at permanently higher levels. This part is due to the rise in the firm's labor demand, explained by other factors than wages. In particular, the fall in the equilibrium real interest rate (which acts as the firm's discount rate) raises the present value of future profits which prompts the firm to expand production, post more vacancies and hire more labor.

We can see these effects in Figure 5. The increase in savings desire after the credit crunch depresses the real interest rate which makes equity more valuable. The expansion in hiring raises private output. We see the overshooting pattern which is characteristic of the credit crunch also

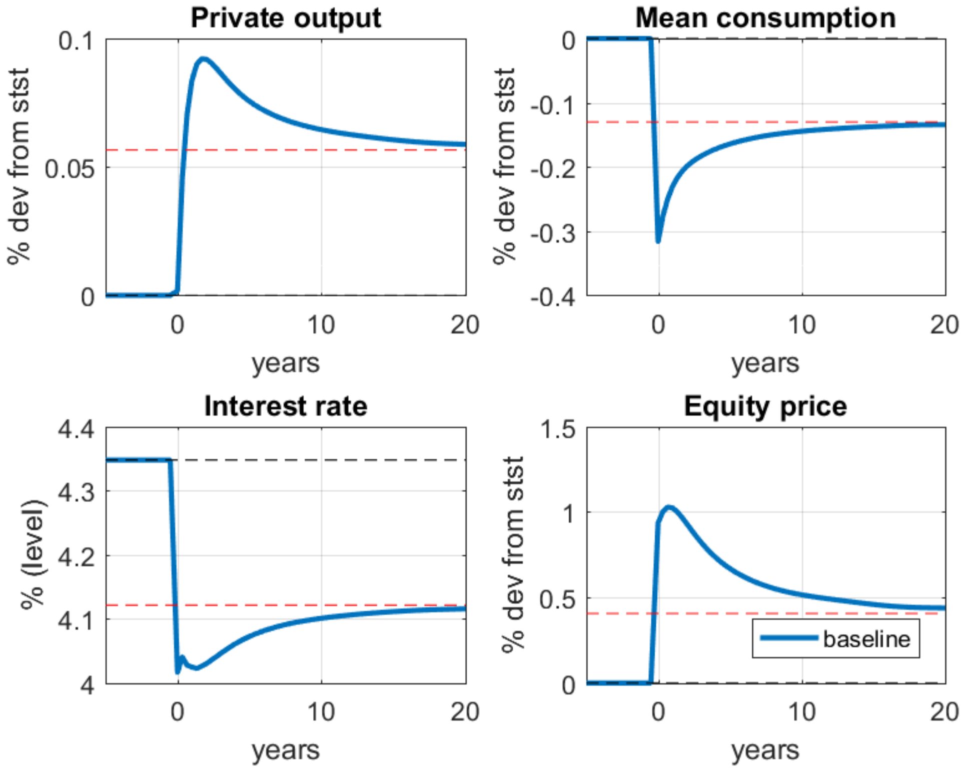


Figure 5: Impulse responses after a tightening of the borrowing constraint – goods and asset markets

in Guerrieri and Lorenzoni (2017). However, unlike in their setup, the credit crunch in our case is expansionary. Recall that both models talk about the supply side of the economy, and ignore the obvious negative effects of the credit crunch on aggregate demand which undoubtedly entails a recession. Given the sign of the *deviations* from a standard model of aggregate demand, we view our results as more in line with characteristics of the post-crisis recovery, i.e. strong rebound in economic activity and employment together with disappointing wage dynamics.

Figure 5 also shows falling average consumption which can be surprising at first sight, given roughly unchanged aggregate savings and higher output. The reason is that higher vacancy posting costs crowd out consumption, despite a rise in production.[22] This is also behind the fact that firm dividends actually fall despite higher revenues and a lower wage bill (resulting in the earlier discussed leftward shift in the asset supply curve). Essentially, the firm reinvests the higher surplus coming from increased profitability into future jobs by posting more vacancies.

[22] In the current model setup there is another reason: in the baseline parametrization, there is no government. This means that instead of unemployment benefits from the government, unemployed households engage in home production which falls as more of them get a job, thereby subtracting from private output. In addition, interest payments on bonds B do not come from the government (financed through taxes) but from abroad, earned on foreign assets. The fall in the interest rate therefore permits a smaller trade deficit $r_t B$, or equivalently, the improving current account also contributes toward crowding out consumption. The reason for ommiting government is that introducing involves some numerical difficulties, but this does not change the main message of our model. The resource constraint is now: $C_t + \xi v_t - r_t B = Y_t + h u_t$

3.4 Sensitivity analysis

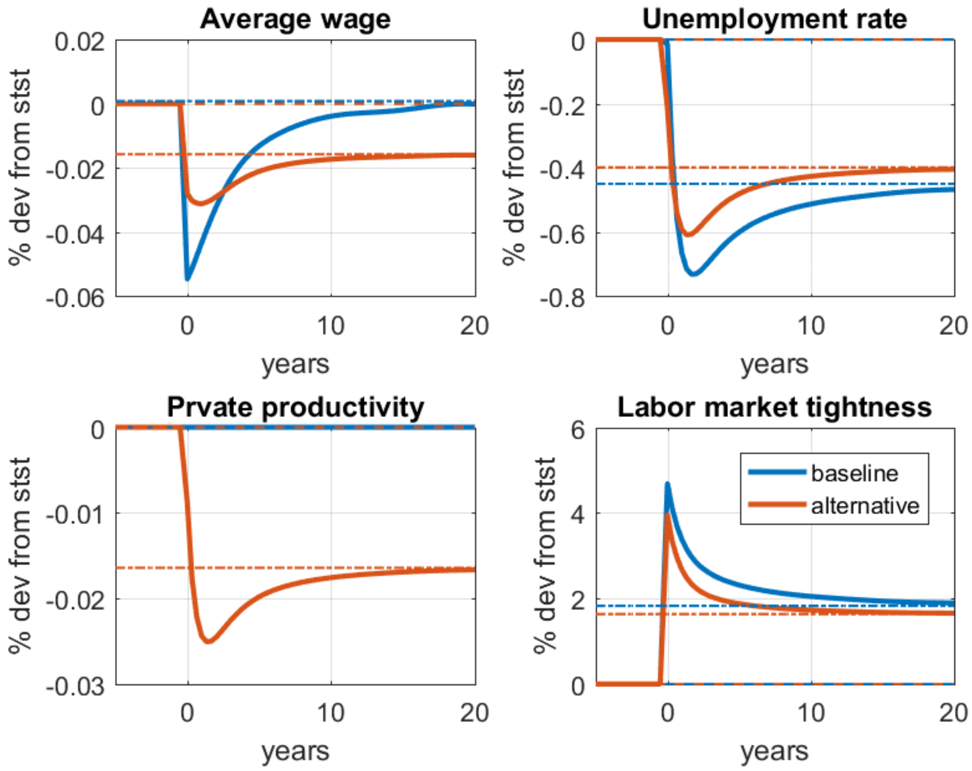


Figure 6: Impulse responses after a tightening of the borrowing constraint – labor market

By introducing a non-linear production technology $\alpha > 0$ we are also able to capture falling productivity following the credit crunch. Decreasing returns to scale in the production function can be interpreted as a modelling shortcut to persistent damage of the supply side of the economy due to a prolonged period of stagnation. In this case lower productivity also results in lower steady state wages.

4 Conclusion

In this paper we have developed a continuous time heterogeneous agent model with search-and-matching frictions in the labor market (HACT+SAM) and analysed the effects of a credit crunch. The combination of labor market frictions and precautionary motives (stemming from incomplete financial markets and endogenous idiosyncratic risk) provides a channel to explain the weak post-crisis wage recovery against the backdrop of tightening labor markets and falling unemployment. As the borrowing constraint tightens for households, they increase their self-insurance attemps: apart from precautionary savings this can be achieved by accepting employment even at lower wages. Allowing for decreasing returns to scale in production we can also capture the lacklustre productivity performance.

Our model ignores nominal rigidities and therefore is unable to capture aggregate demand effects which are undoubtedly an important part of business cycle fluctuations in general, and credit crunch scenarios in particular. Nevertheless, our paper points out a channel which can contribute to a deeper understanding of post-crisis wage dynamics and can optentially explain the weaker positive co-movement of wages and aggregate demand observed in the data. That said, introducing nominal rigidities is high on our research agenda, as well as accounting for an endogenous labor force participation choice.

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A Numerical algorithm

Based in large part on Bardóczy (2017), but without capital (and iterating instead over the interest rate). Details of the upwinding scheme and finite difference method used in HACT models are described by Achdou et al. (2017), with special attention to their Online Appendix. The credit crunch (tightening of the borrowing constraint) is incorporated into the upwinding scheme similarly as suggested by Mellior (2016).

A.1 Stationary equilibrium

Set up a discrete grid for $a \in \{a_i\}_{i=1}^{I}$

1. start iterating over $\ell = 1, 2, \ldots$ – outer loop
2. guess tightness θ^1 – update $\theta^{\ell+1}$ based on (14) in step 8

   using the matching function $M(u, v) = \chi u^{\eta} v^{1-\eta}$, and the definition $\theta = \frac{v}{u}$

$$\text{(1)} \qquad \lambda^f = \chi \theta^{-\eta}$$

$$\text{(2)} \qquad \lambda^w = \chi \theta^{1-\eta}$$

$$\text{(3)} \qquad u = \frac{\sigma}{\sigma + \lambda^w}$$

$$v = \theta\, u$$

3. guess the interest rate $r^1 = 0.9\,\rho$ – update $r^{\ell+1}$ based on (19) in step 9 [23]

$$\text{(18)} \qquad T = r\,B + u\,h$$

4. guess wage schedule $\omega^1(a_i) = \omega_i = \beta[z(1-u)^{-\alpha}]$ – update $\omega^{\ell+1}(a_i)$ based on (17) in step 10

$$y_{is} = \begin{cases} \omega_i - T & \text{if } s = s^e \\ h - T = (1-u)h - rB & \text{if } s = s^u \end{cases}$$

$$\text{(15)} \qquad \pi(a_i) = \pi_i = z(1-u)^{-\alpha} - \omega_i$$

5. solve the worker's problem – first inner loop over $\imath$
   (a) iterations $\imath = 1, 2, \ldots$ over the worker's HJB: guess $\mathbf{W}^1 = \left\{ \frac{u\left(y_{is} + r\,a_i\right)}{\rho} \right\}_{\forall is}$ – update $W^{\imath+1}$ at the end of step 5c)

[23] ρ is the complete market interest rate. With incomplete markets there will be extra precautionary saving motive, and the incomplete market interest rate will be depressed downwards.

(b) discretize the HJB equation (6), (7) over the state space $\{a_i\} \times \{s^e, s^u\}$ by the finite difference method and the upwinding scheme (also using (8), (9)). The state constraints are appended depending on how tighter the new borrowing constraint is than the original one ($x \geq 1$) – build the $\mathbf{A}^\imath$ matrix[24]

$$\partial_a W_{is}^F \equiv \begin{cases} u'(y_{Is} + r\,a_I) & \text{for } i = I \\ \frac{W_{i+1,s} - W_{is}}{\Delta a} & \text{otherwise} \end{cases} \qquad \partial_a W_{is}^B \equiv \begin{cases} \frac{W_{is} - W_{i-1,s}}{\Delta a} & \text{otherwise} \\ u'(y_{is} + r\,a_i - (x-i)\Delta a) & \text{for } i \leq x \end{cases}$$

$$c_{is}^F = u'^{-1}\left(\partial_a W_{is}^F\right) \qquad \dot{a}_{is}^F = y_{is} + r\,a_i - c_{is}^F$$

$$c_{is}^B = u'^{-1}\left(\partial_a W_{is}^B\right) \qquad \dot{a}_{is}^B = y_{is} + r\,a_i - c_{is}^B$$

$$c_{is}^0 = y_{is} + r\,a_i - (x - \min\{i, x\})\Delta a$$

$$c_{is} = \mathbb{I}_{\{0<\dot{a}_{is}^F\}}\, c_{is}^F + \mathbb{I}_{\{\dot{a}_{is}^B<0\}}\, c_{is}^B + \mathbb{I}_{\{\dot{a}_{is}^F \leq 0 \leq \dot{a}_{is}^B\}}\, c_{is}^0$$

$$\dot{a}_{is} = y_{is} + r\,a_i - c_{is}$$

$$\rho W_{is} = u(c_{is}) + \frac{W_{i+1,s} - W_{is}}{\Delta a}\,[\dot{a}_{is}]^+ + \frac{W_{is} - W_{i-1,s}}{\Delta a}\,[\dot{a}_{is}]^- + \sum_{s' \neq s} \lambda_{ss'}\left[W_{is'} - W_{is}\right]$$

$$\rho W_{is} = u(c_{is}) - \frac{[\dot{a}_{is}]^-}{\Delta a} W_{i-1,s} + \left(\frac{[\dot{a}_{is}]^- - [\dot{a}_{is}]^+}{\Delta a}\right) W_{is} + \frac{[\dot{a}_{is}]^+}{\Delta a} W_{i+1,s} + \sum_{s' \neq s} \lambda_{ss'}\left[W_{is'} - W_{is}\right]$$

$$\rho\,\mathbf{W}^\imath = \mathbf{u}(\mathbf{W}^\imath) + \mathbf{A}(\mathbf{W}^\imath; \mathbf{r})\,\mathbf{W}^\imath$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_e} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_u} \end{bmatrix} + \begin{bmatrix} -\sigma\,\mathbf{I} & \sigma\,\mathbf{I} \\ \lambda^w\,\mathbf{I} & -\lambda^w\,\mathbf{I} \end{bmatrix}; \quad \mathbf{A_s} = \begin{bmatrix} \frac{\dot{a}_{1s}^- - \dot{a}_{1s}^+}{\Delta a} & \frac{\dot{a}_{1s}^+}{\Delta a} & 0 & 0 & \cdots & 0 \\ -\frac{\dot{a}_{2s}^-}{\Delta a} & \frac{\dot{a}_{2s}^- - \dot{a}_{2s}^+}{\Delta a} & \frac{\dot{a}_{2s}^+}{\Delta a} & 0 & \cdots & 0 \\ 0 & -\frac{\dot{a}_{3s}^-}{\Delta a} & \frac{\dot{a}_{3s}^- - \dot{a}_{3s}^+}{\Delta a} & \frac{\dot{a}_{3s}^+}{\Delta a} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\frac{\dot{a}_{Is}^-}{\Delta a} & \frac{\dot{a}_{Is}^- - \dot{a}_{Is}^+}{\Delta a} \end{bmatrix}$$

(c) update the value function $\mathbf{W}^{\imath+1}$ using the implicit method, and go back to step 5b)

$$\frac{\mathbf{W}^{\imath+1} - \mathbf{W}^\imath}{\Delta} + \rho\,\mathbf{W}^{\imath+1} = \mathbf{u}^\imath + \mathbf{A}^\imath\,\mathbf{W}^{\imath+1}$$

$$\mathbf{W}^{\imath+1} = \left[\left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^\imath\right]^{-1}\left(\mathbf{u}^\imath + \frac{1}{\Delta}\mathbf{W}^\imath\right)$$

(d) after convergence save optimal value funtion, and consumption and savings policies (and keep the final $\mathbf{A}$ matrix):

$$\mathbf{W} = \mathbf{W}^\imath \qquad c_{is} = c_{is}^\imath \qquad \dot{a}_{is} = \dot{a}_{is}^\imath$$

[24]The terms in the $\mathbf{A}$ matrix are defined as $[\dot{a}_{is}]^+ = \max\{\dot{a}_{is}, 0\}$ and $[\dot{a}_{is}]^- = \min\{\dot{a}_{is}, 0\}$. This is in contrast to Mellior (2016) and Achdou et al. (2017) where it is defined as $[\dot{a}_{is}]^+ = \max\{\dot{a}_{is}^F, 0\}$. This difference does not matter for $x = 1$ (no tightening in the borrowing constraint), but it does for any other $x > 1$. Under the definition of Mellior (2016) the $\mathbf{A_s}$ matrix would only contain zeros in the rows corresponding to the inadmissible region $i < x$ whenever $\dot{a}_{is}^F < 0 < \dot{a}_{is}^B = (x-i)\Delta a$ while there we require positive actual savings $\dot{a}_{is} > 0$.

6. calculate stationary distribution

   it is basically an eigenvalue problem (imposing $\sum_s \sum_i g_{is} = 1$), solving the discretized KF equation which involves the same $\mathbf{A}$ matrix as the discretized HJB equation

$$(10),(11) \qquad 0 = \mathbf{A}^T \mathbf{g}$$

7. solve the firm's problem – second inner loop over $\jmath$

   (a) discretize the job HJB equation (13) over the state space $\{a_i\}_{i=1}^I$ by the finite difference method and the upwinding scheme – ending up with the <u>*already calculated*</u> $\mathbf{A_e}$ matrix (saving is exogenous to the firm and we already applied the upwinding scheme for the household)

$$(13) \qquad (\sigma + r)\, J_i = \pi_i + \frac{J_{i+1} - J_i}{\Delta a}\, [\dot{a}_{ie}]^+ + \frac{J_i - J_{i-1}}{\Delta a}\, [\dot{a}_{ie}]^-$$

$$(\sigma + r)\, J_i = \pi_i - \frac{[\dot{a}_{ie}]^-}{\Delta a} J_{i-1} + \left(\frac{[\dot{a}_{ie}]^- - [\dot{a}_{ie}]^+}{\Delta a} \right) J_i + \frac{[\dot{a}_{ie}]^+}{\Delta a} J_{i+1}$$

$$(\sigma + r)\, \mathbf{J} = \pi + \mathbf{A_e}\, \mathbf{J}$$

   (b) iterations $\jmath = 1, 2, \dots$ over the firm's job HJB: guess $\mathbf{J}^1 = \left\{ \frac{\pi(a_i)}{\sigma + r} \right\}_{\forall i}$ – update $J^{\imath+1}$ at the end of step 7c)

   (c) update $\mathbf{J}^{\jmath+1}$ according to the implicit method, and go back to step 7b)

$$\frac{\mathbf{J}^{\jmath+1} - \mathbf{J}^{\jmath}}{\Delta} + (\sigma + r)\mathbf{J}^{\jmath+1} = \pi^{\jmath} + \mathbf{A_e}\, \mathbf{J}^{\jmath+1}$$

$$\mathbf{J}^{\jmath+1} = \left[\left(\frac{1}{\Delta} + \sigma + r \right) \mathbf{I} - \mathbf{A_e} \right]^{-1} \left(\pi^{\jmath} + \frac{1}{\Delta} \mathbf{J}^{\jmath} \right)$$

   (d) after convergence save the optimal value function $\mathbf{J} = \mathbf{J}^{\jmath}$

8. evaluate free entry condition (vacancy decision)

$$(14) \qquad FE = -\xi + \lambda^f\, \mathbf{J}^T \frac{\mathbf{g_u}}{u} \Delta a$$

   with $FE > 0$ firms are too profitable, so update with a higher guess for labor market tightness (in the outer loop)

$$\theta^{\ell+1} = \theta^{\ell} + \Delta\theta \cdot FE^{\ell}$$

9. check asset market clearing

$$(16) \qquad d = \pi^T\, \mathbf{g_e}\, \Delta a - \xi v$$

$$(5) \qquad p = \frac{d}{r}$$

$$(19) \qquad AD = \mathbf{a}^T [\mathbf{g_e} + \mathbf{g_u}]\, \Delta a - B - p$$

with $AD > 0$ there is an excess demand for assets (equivalently, too much saving), so the interest rate should be decreased. Update with bisection:

$$r^{\ell+1} = \begin{cases} \frac{r^\ell + r^\ell_{min}}{2} & \text{if } AD^\ell > 0 \\ \frac{r^\ell + r^\ell_{max}}{2} & \text{if } AD^\ell < 0 \\ r^\ell & \text{if } AD^\ell \approx 0 \end{cases} \qquad r^{\ell+1}_{min} = \begin{cases} r^\ell_{min} & \text{if } AD^\ell > 0 \\ r^\ell & \text{if } AD^\ell < 0 \end{cases} \qquad r^{\ell+1}_{max} = \begin{cases} r^\ell & \text{if } AD^\ell > 0 \\ r^\ell_{max} & \text{if } AD^\ell < 0 \end{cases}$$

10. update the wage schedule $\omega^{\ell+1}(a_i)$ from the bargaining equation (17)
11. use the updates
    - $\theta^{\ell+1}$ based on step 8,
    - $r^{\ell+1}$ based on step 9,
    - $\omega^{\ell+1}(a_i)$ based on step 10,

    and go back to step 2 – stop if both AD and FE are small enough

A.2 Transition dynamics

- embed the iterations for the HJB and KFE equations along the time path n into an outer loop ℓ over a triplet of guesses $(r_n, \theta_n, \hat{p}_1)\ \forall n$
  - iterate HJB equations *backward* in time
  - iterate KFE equations *forward* in time (previously did not iterate)
  - keep all steps, as they are now correspond to the time path
- equity price on impact $\hat{p}_1$ needs to be guessed as the unexpected shock changes future profits and revalues the assets

1. discretize the state space
   - the employment state is already discrete $s \in \{s^e, s^u\}$
   - asset grid $a \in \{a_i\}_{i=1}^I$ with Δa
   - time grid $t \in \{t_n\}_{n=1}^N$ with $\{\Delta t_n\}_{n=1}^{N-1}$ non-uniform steps
2. set initial and terminal conditions $\mathbf{g_1}, \mathbf{W_N}, \mathbf{J_N}, \omega_N(a_i)$ (e.g. potentially different stationary equilibria)
3. specify exogenous time path for TFP process

$$z_n = 1 + (z_0 - 1)\ e^{-\nu t_n}$$

4. start iterating over $\ell = 1, 2, \ldots$ – outer loop
5. guess time path $\forall n$ for tightness θ_n^1 – update $\theta_n^{\ell+1}$ from the final step

$$\lambda_n^f = \chi \theta_n^{-\eta}$$
$$\lambda_n^w = \chi \theta_n^{1-\eta}$$

   unemployment is recovered by iterating *forward* on the differential equation (3). Both implicit and explicit iterations work but explicit is more stable (and maintains no change on impact for unemployment, which is a state variable).

$$\text{expl:} \quad \frac{u_{n+1} - u_n}{\Delta t_n} = \sigma - (\lambda_n^w + \sigma) u_n \qquad u_{n+1} = \Delta t_n\, \sigma + \Big[1 - \Delta t_n\, (\lambda_n^w + \sigma)\Big] u_n$$

$$\text{impl:} \quad \frac{u_{n+1} - u_n}{\Delta t_n} = \sigma - (\lambda_{n+1}^w + \sigma) u_{n+1} \qquad u_{n+1} = \frac{u_n + \Delta t_n\, \sigma}{1 + \Delta t_n\, (\lambda_{n+1}^w + \sigma)}$$

6. guess time path $\forall n$ for the interest rate r_n^1 – update $r_n^{\ell+1}$ from the final step

$$T_n = r_n B + u_n\ h$$

7. solve the worker's <u>and</u> firm's problem *simultaneously* – first inner loop *backward* over n

(a) iterations $n = N, N-1, N-2, \dots 1$: Instead of initial guesses, start from the terminal conditions $\mathbf{W}_N, \mathbf{J}_N,\ \omega_N(a_i)$ - and keep all steps! Not until convergence, but until reaching t_1.

$$y_{is}^n = \begin{cases} \omega_n(a_i) - T_n & \text{if } s = s^e \\ h - T_n = (1-u_n)h - r_n B & \text{if } s = s^u \end{cases}$$

$$\pi_n(a_i) = \pi_{i,n} = z_n(1-u_n)^{-\alpha} - \omega_n(a_i)$$

(b) from the worker's discretized time-dependent HJB equation build the time-dependent $\mathbf{A}_n$ matrix (for $n = N$ this should just give back the terminal $\mathbf{A}_N$)

$$\partial_a W_{is}^{F,n} \equiv \begin{cases} u'(y_{Is}^n + r^n\, a_I) & \text{for } i = I \\ \frac{W_{i+1,s}^n - W_{is}^n}{\Delta a} & \text{otherwise} \end{cases} \qquad \partial_a W_{is}^{B,n} \equiv \begin{cases} \frac{W_{is}^n - W_{i-1,s}^n}{\Delta a} & \text{otherwise} \\ u'(y_{is}^n + r^n\, a_i) & \text{for } i = x \\ u'(y_{is}^n + r^n\, a_i - \Delta a) & \text{for } i < x \end{cases}$$

$$c_{is}^{F,n} = u'^{-1}\Big(\partial_a W_{is}^{F,n}\Big) \qquad \dot{a}_{is}^{F,n} = y_{is}^n + r^n\, a_i - c_{is}^{F,n}$$

$$c_{is}^{B,n} = u'^{-1}\Big(\partial_a W_{is}^{B,n}\Big) \qquad \dot{a}_{is}^{B,n} = y_{is}^n + r^n\, a_i - c_{is}^{B,n}$$

$$c_{is}^{0,n} = y_{is}^n + r^n\, a_i - \mathbb{I}_{\{i<x\}}\Delta a$$

$$c_{is}^n = \mathbb{I}_{\{0<\dot{a}_{is}^{F,n}\}}\, c_{is}^{F,n} + \mathbb{I}_{\{\dot{a}_{is}^{B,n}<0\}}\, c_{is}^{B,n} + \mathbb{I}_{\{\dot{a}_{is}^{F,n}\le 0 \le \dot{a}_{is}^{B,n}\}}\, c_{is}^{0,n}$$

$$\dot{a}_{is}^n = y_{is}^n + r^n\, a_i - c_{is}^n$$

$$\rho W_{is}^n = u(c_{is}^n) + \frac{W_{i+1,s}^n - W_{is}^n}{\Delta a}\,[\dot{a}_{is}^n]^+ + \frac{W_{is}^n - W_{i-1,s}^n}{\Delta a}\,[\dot{a}_{is}^n]^- + \sum_{s'\neq s} \lambda_{ss'}^n \Big[W_{is'}^n - W_{is}^n\Big] + \frac{W_{is}^{n+1} - W_{is}^n}{\Delta t_n}$$

$$\rho W_{is}^n = u(c_{is}^n) - \frac{[\dot{a}_{is}^n]^-}{\Delta a} W_{i-1,s}^n + \left(\frac{[\dot{a}_{is}^n]^- - [\dot{a}_{is}^n]^+}{\Delta a}\right) W_{is}^n + \frac{[\dot{a}_{is}^n]^+}{\Delta a} W_{i+1,s}^n + \sum_{s'\neq s} \lambda_{ss'}^n \Big[W_{is'}^n - W_{is}^n\Big] + \frac{W_i}{}$$

$$\rho\, \mathbf{W}_n = \mathbf{u}(\mathbf{W_n}) + \mathbf{A}(\mathbf{W_n};\, \mathbf{r_n})\, \mathbf{W}_n + \frac{\mathbf{W}_{n+1} - \mathbf{W}_n}{\Delta t_n}$$

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{A_{e,n}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{u,n}} \end{bmatrix} + \begin{bmatrix} -\sigma\, \mathbf{I} & \sigma\, \mathbf{I} \\ \lambda_n^w\, \mathbf{I} & -\lambda_n^w\, \mathbf{I} \end{bmatrix};$$

$$\mathbf{A_{s,n}} = \begin{bmatrix} \frac{[\dot{a}_{1s}^n]^- - [\dot{a}_{1s}^n]^+}{\Delta a} & \frac{[\dot{a}_{1s}^n]^+}{\Delta a} & 0 & 0 & \cdots & 0 \\ -\frac{[\dot{a}_{2s}^n]^-}{\Delta a} & \frac{[\dot{a}_{2s}^n]^- - [\dot{a}_{2s}^n]^+}{\Delta a} & \frac{[\dot{a}_{2s}^n]^+}{\Delta a} & 0 & \cdots & 0 \\ 0 & -\frac{[\dot{a}_{3s}^n]^-}{\Delta a} & \frac{[\dot{a}_{3s}^n]^- - [\dot{a}_{3s}^n]^+}{\Delta a} & \frac{[\dot{a}_{3s}^n]^+}{\Delta a} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\frac{[\dot{a}_{Is}^n]^-}{\Delta a} & \frac{[\dot{a}_{Is}^n]^- - [\dot{a}_{Is}^n]^+}{\Delta a} \end{bmatrix}$$

(c) calculate the emphprevious period's value function $\mathbf{W}_{n-1}$ from the worker's HJB equations, using the implicit method

$$\rho\,\mathbf{W}_{n-1} = \mathbf{u}_n + \mathbf{A}_n\,\mathbf{W}_{n-1} + \frac{\mathbf{W}_n - \mathbf{W}_{n-1}}{\Delta t_{n-1}}$$

$$\left[\left(\frac{1}{\Delta t_{n-1}} + \rho\right)\mathbf{I} - \mathbf{A}_n\right]\mathbf{W}_{n-1} = \mathbf{u}_n + \frac{1}{\Delta t_{n-1}}\mathbf{W}_n$$

(d) calculate the previous period's value function $\mathbf{J}_{n-1}$ from the firm's HJB equations, using the implicit method

$$(\sigma + r_n)\mathbf{J}_{n-1} = \pi_n + \mathbf{A}_{\mathbf{e},\mathbf{n}}\,\mathbf{J}_{n-1} + \frac{\mathbf{J}_n - \mathbf{J}_{n-1}}{\Delta t_{n-1}}$$

$$\left[\left(\frac{1}{\Delta t_{n-1}} + \sigma + r_n\right)\mathbf{I} - \mathbf{A}_{\mathbf{e},\mathbf{n}}\right]\mathbf{J}_{n-1} = \pi_n + \frac{1}{\Delta t_{n-1}}\mathbf{J}_n$$

(e) given the worker and firm value functions $\mathbf{W}_{n-1}, \mathbf{J}_{n-1}$, calculate the previous period's wage schedule $\omega_{n-1}(a_i)$ from the bargaining equation (Nash or egalitarian)

(f) at each step n (!) keep the original value functions and optimal policies as well as the wage schedule, the profit and the $\mathbf{A}$ matrix

$$\mathbf{W}_n \quad \mathbf{J}_n \quad \omega_n(a_i) \quad c_{is}^n \quad \dot{a}_{is}^n \quad \pi_n(a_i) \quad \mathbf{A}_n$$

(g) go back to step 7a) with the freshly calculated $\mathbf{W}_{n-1} \quad \mathbf{J}_{n-1} \quad \omega_{n-1}(a_i)$ and start the $n-1$-st step, until reaching $n = 1$

8. guess impact equity price $\hat{p}_1^1$ to revalue assets – update $p_1^{\ell+1}$ from the final step

- on impact p_1 from the initial condition jumps to $\hat{p}_1$ as a result of the shock – due to change to the firm's future profits
- each agent sees their assets revalued on impact, without being able to do anything

$$\hat{a}_i = a_i\left[1 + \frac{\hat{p}_1 - p_1}{B + p_1}\right] \qquad \hat{g}_1(\hat{a}_i, s) = g_1(a_i, s)$$

so the distribution shifts to a rescaled grid, but stays the same

- we want to work with the original grid $\{a_i\}$, so we interpolate the new distribution onto the original grid

$$\hat{g}_1(a_i, s) = \texttt{pchip}\Big(\hat{a}_i, \underbrace{\hat{g}_1(\hat{a}_i, s)}_{g_1(a_i,s)};\ a_i\Big)$$

- $\hat{\mathbf{g}}_1$ needs to integrate to one, so some rescaling might be needed

9. calculate time path for the distribution – second inner loop *forward* over n

(a) iterations $n = 1, 2, 3, \ldots N-1$: start from the <u>*revalued* initial</u> distribution $\hat{\mathbf{g}}_1$ - and keep all steps! Not until convergence, but until reaching t_N.

(b) calculate next period's distribution $\mathbf{g}_{n+1}$ from the discretized KF equation, involving the same $\mathbf{A}_n$ matrix (iterating using the implicit method)

$$\frac{\mathbf{g}_{n+1} - \mathbf{g}_n}{\Delta t_n} = \mathbf{A}_n^T \, \mathbf{g}_{n+1}$$
$$\mathbf{g}_{n+1} = \Big[\mathbf{I} - \Delta t_n \, \mathbf{A}_n^T\Big]^{-1} \, \mathbf{g}_n$$

(c) at each step keep the original distribution $\mathbf{g}_n$ and go back to step 9a) with the freshly calculated $\mathbf{g}_{n+1}$ and start the $n+1$-st step, until reaching $n = N$

10. calculate time path for vacancies (and check unemployment!)

$$u_n = \sum_{i=1}^{I} \sum_{s=s^u} g_{is}^n \, \Delta a$$
$$v_n = \theta_n \, u_n$$

we already have u_n from iterating on the law of motion (3) from step 5. But for large $\{\Delta t_n\}$ integrating from the distribution instead is more precise, so unless $\alpha \neq 0$, it is better to use this approach.

11. evaluate free entry condition

$$FE_n = -\xi + \lambda_n^f \, \mathbf{J}_n^T \, \frac{\mathbf{g_u}_n}{u_n} \, \Delta a$$

12. check asset market clearing

we also need the time path for the equity price (iterated implicitly backwards from terminal p_N based on no arbitrage $p_n \, r_n = d_n + \frac{p_{n+1} - p_n}{\Delta t_n}$) – this will lead to a $\tilde{p}_1$

$$d_n = \pi_n^T \, \mathbf{g_e}_n \, \Delta a - \xi \, v_n$$
$$p_{n-1} = \left[\frac{1}{\Delta t_{n-1}} + r_{n-1}\right]^{-1} \left[d_{n-1} + \frac{p_n}{\Delta t_{n-1}}\right]$$
$$AD_n = \mathbf{a}^T \big[\mathbf{g_e}_n + \mathbf{g_u}_n\big] \, \Delta a - B - p_n$$

13. use the updates

with decreasing sequences $\{\Delta_{rn}, \Delta_{\theta n}\}_n$ to help convergence

$$\hat{p}_1^{\ell+1} = \Delta_p \, \tilde{p}_1^{\ell} + (1 - \Delta_p) \, \hat{p}_1^{\ell}$$
$$r_n^{\ell+1} = r_n^{\ell} - \Delta_{rn} \cdot \Delta AD_n^{\ell} \qquad \forall n$$
$$\theta_n^{\ell+1} = \theta_n^{\ell} + \Delta_{\theta n} \cdot FE_n^{\ell} \qquad \forall n$$

where ΔAD_n^{ℓ} is alternating forward or backward difference.

Then go back to step 5 – stop if all elements in the triplet $\Big[\theta_n, r_n, \hat{p}_1, \Big]$ have converged

B Additional figures

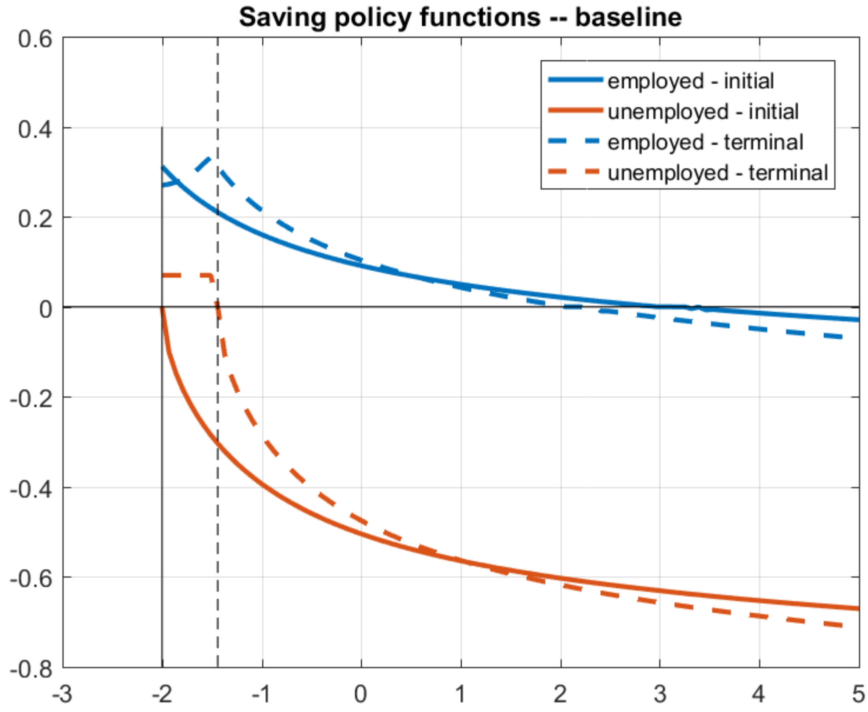

Figure 7: Savings policy functions

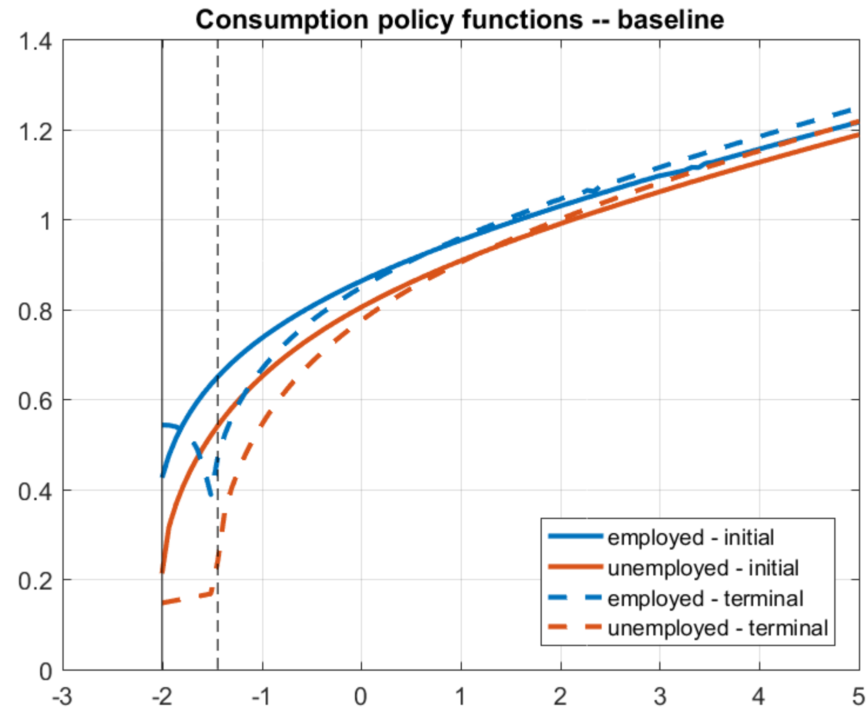

Figure 8: Consumption policy functions